The period function of a delay differential equation and an application[1]

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Abstract

We consider the delay differential equation $\dot{x}(t) = -\mu x(t) + f(x(t-\tau))$, where μ, τ are positive parameters and f is a strictly monotone, nonlinear C^1-function satisfying $f(0) = 0$ and some convexity properties. It is well known that for prescribed oscillation frequencies (characterized by the values of a discrete Lyapunov functional) there exists $\tau^* > 0$ such that for every $\tau > \tau^*$ there is a unique periodic solution. The period function is the minimal period of the unique periodic solution as a function of $\tau > \tau^*$. First we show that it is a monotone nondecreasing Lipschitz continuous function of τ with Lipschitz constant 2. As an application of our theorem we give a new proof of some recent results of Yi, Chen and Wu [14] about uniqueness and existence of periodic solutions of a system of delay differential equations.

1 Introduction

Consider the delay differential equation

$$\dot{x}(t) = -\mu x(t) + \delta f(x(t-\tau)), \tag{1}$$

where $\mu > 0$, $\delta \in \{-1, 1\}$ and $f\colon \mathbb{R} \to \mathbb{R}$ is a strictly increasing, odd C^1-function such that the function $\xi \mapsto \frac{\xi f'(\xi)}{f(\xi)}$ is strictly monotone decreasing on $(0, \infty)$ and there exists $\xi^* \in (0, \infty)$ such that $f(\xi^*) = \mu\xi^*$. These hypotheses are assumed throughout the paper. The cases $\delta = -1$ and $\delta = 1$ are regarded as the negative and positive feedback cases, respectively. This equation is often applied in models of neural networks and time delay appears due to finite conduction velocities or synaptic transmission. In this topic the feedback functions of type $f(\xi) = \alpha \arctan(\beta\xi)$ are widely applied and satisfy the assumptions above. For a general overview of neural networks we refer the reader to [15].

In neural systems, periodic solutions are of great importance. From the papers of Cao, Krisztin and Walther [1, 4, 5, 6] and from the monograph of Krisztin, Walther and Wu [7] we get a very detailed and clear picture of the periodic orbits of equation (1). For our purposes, their most important result is that they have proven that if the feedback is positive, then there exists a threshold $\tau^* > 0$ such that for every $\tau > \tau^*$ there exists, up to time translation, exactly one periodic solution of equation (1), which changes signs at least once and at most twice on any interval with length τ.

In Section 3 we investigate the minimal period of this slowly oscillating periodic solution, as a function of τ, and in Theorem 3.2 we prove that this function is a monotone nondecreasing Lipschitz continuous

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function with Lipschitz constant 2. The proof is based on Proposition 3.1, which can be intuitively formulated as follows: "the greater the parameter τ is, the greater amplitude the slowly oscillating periodic solution has".

In Section 4 we investigate the following system of delay differential equations:

$$\begin{aligned} \dot{x}^0(t) &= -\mu x^0(t) + f(x^1(t)), \\ &\vdots \\ \dot{x}^{n-1}(t) &= -\mu x^{n-1}(t) + f(x^n(t)), \\ \dot{x}^n(t) &= -\mu x^n(t) + \delta f(x^0(t-1)), \end{aligned} \tag{2}$$

where μ, δ and f have the same properties as in equation (1) and $2 \leq n \in \mathbb{N}$. This can be a model of a unidirectional ring of interacting neurons. Based on Theorem 3.2 and some ideas of Yi, Chen and Wu [14] we obtain results on uniqueness and absence of periodic solutions. In the paper [14], the results are only proven for the case of positive feedback and $n = 2$. We generalize these results for the case of negative feedback and $n \geq 3$, too (case $n = 1$ is worked out by Chen and Wu [2]). In terms of the period function, we also reformulate their conjecture concerning uniqueness and absence of "relatively slowly oscillating" periodic orbits of (2), and we hope that this formulation may be the key to the proof of the conjecture.

2 Preliminaries

First of all, let us clarify some notations. $\mathbb{R}, \mathbb{R}_+$ and $\mathbb{N}$ denote the set of reals, nonnegative reals and nonnegative integers, respectively. If $A, B \subset \mathbb{R}$, then $C(A, B)$ denotes the set of real-valued functions with domain A and range contained in B. C^1 denotes the set of continuously differentiable real-valued functions with domain $\mathbb{R}$. For a simple closed curve c, $\mathrm{int}(c), \mathrm{ext}(c)$ and $|c|$ denote the interior, exterior and the trace of c, respectively. Now, in this section we recall some earlier definitions and results.

The natural phase space for (1) is $C([-\tau, 0], \mathbb{R})$ equipped with the maximum norm. Let this be denoted in the sequel by C_τ. If x is a solution of (1) on some interval, then we let $x_{t,\tau} \in C_\tau$ be defined by

$$x_{t,\tau}(\theta) = x(t+\theta) \quad \text{for all} \quad \theta \in [-\tau, 0],$$

at least where it makes sense.

According to Mallet-Paret and Sell [11], we define discrete Lyapunov functionals in order to characterize the periodic solutions of (1). For every $\tau > 0$ let

$$V_\tau^+ \colon C_\tau \setminus \{0\} \to \{0, 2, 4, \ldots, \infty\}, \qquad V_\tau^- \colon C_\tau \setminus \{0\} \to \{1, 3, 5, \ldots, \infty\},$$

be defined as follows:

$$V_\tau^+(\varphi) = \begin{cases} \mathrm{sc}(\varphi, [-\tau, 0]), & \text{if } \mathrm{sc}(\varphi, [-\tau, 0]) \text{ is even or infinite,} \\ \mathrm{sc}(\varphi, [-\tau, 0]) + 1, & \text{if } \mathrm{sc}(\varphi, [-\tau, 0]) \text{ is odd,} \end{cases}$$

$$V_\tau^-(\varphi) = \begin{cases} \mathrm{sc}(\varphi, [-\tau, 0]), & \text{if } \mathrm{sc}(\varphi, [-\tau, 0]) \text{ is odd or infinite,} \\ \mathrm{sc}(\varphi, [-\tau, 0]) + 1, & \text{if } \mathrm{sc}(\varphi, [-\tau, 0]) \text{ is even,} \end{cases}$$

where $\mathrm{sc}\colon C_\tau \to \mathbb{N}$ is defined as follows:

$$\mathrm{sc}(\varphi) = \sup\{k \in \mathbb{N} : \quad \text{there is a strictly increasing finite sequence } (s^i)_0^k \subset [-\tau, 0] \\ \text{with } \varphi(s^{i-1})\varphi(s^i) < 0 \text{ for all } i \in \{1, 2, \ldots, k\}\ \} \leq \infty.$$

By definition, let $\sup \emptyset = 0$.

Definition 2.1. We say that $X\colon [0, T_x] \ni t \mapsto (x(t), \dot{x}(t)) \in \mathbb{R}^2$ is the *D-trajectory* of the periodic C^1-function x, where T_x denotes the minimal period of x.

The proposition below follows from general results of [12] and contains a list of some very important properties of periodic solutions of (1).

Proposition 2.2. *Let $x\colon \mathbb{R} \to \mathbb{R}$ be a nonconstant periodic solution of* (1) *with minimal period $T_x > 0$, and let X denote the D-trajectory of x. Then the following statements are true.*

(i) (i) X is a simple closed curve, and if x has a zero, then $0 \in \mathrm{int}(X)$.

(ii) There are $t_0 \in \mathbb{R}$ and $t_1 \in (t_0, t_0 + T_x)$ such that $0 < \dot{x}(t)$ for all $t_0 < t < t_1$, $x(\mathbb{R}) = [x(t_0), x(t_1)]$ and $\dot{x}(t) < 0$ for all $t_1 < t < t_0 + T_x$.

(iii) If x has a zero, then $x(t + \frac{T_x}{2}) = -x(t)$ for all $t \in \mathbb{R}$.

(iv) (iv) There exists $k \in \mathbb{N}$ such that

$$V_\tau^+(x_{t,\tau}) = 2k \quad \textit{for all} \quad t \in \mathbb{R} \quad \textit{if} \quad \delta = 1,$$

$$V_\tau^-(x_{t,\tau}) = 2k + 1 \quad \textit{for all} \quad t \in \mathbb{R} \quad \textit{if} \quad \delta = -1.$$

According to Proposition 2.2 iv, whenever x is a solution of (1) we shall write $V_\tau^\pm(x)$ instead of $V_\tau^\pm(x_{t,\tau})$ for all $t \in \mathbb{R}$.

Proposition 2.3. *Let $\tau \geq 1$ and $\delta \in \{-1, 1\}$. Let an odd C^1-function $g\colon \mathbb{R} \to \mathbb{R}$ be given which satisfies $g'(0) = f'(0)$ and*

$$g(\xi) > f(\xi) \quad \textit{and} \quad \frac{g'(\xi)}{g(\xi)} > \frac{f'(\xi)}{f(\xi)} \quad \textit{for all} \quad \xi > 0.$$

Let x and z be nonconstant periodic solutions of $\dot{x}(t) = -\mu x(t) + \delta f(x(t-1))$ and $\dot{z}(t) = -\mu x(t) + \delta g(z(t-\tau))$, respectively, with minimal periods $T_x > 0$ and $T_z > 0$. Let X and Z denote the D-trajectories of x and z, respectively. In the case $\delta = 1$, suppose that $V_1^+(x) = V_\tau^+(z) \geq 2$. If $\delta = -1$, suppose that $V_1^-(x) = V_\tau^-(z)$. Then the following situation cannot occur:

$$|Z| \subset |X| \cup \mathrm{ext}(X), \qquad |Z| \cap |X| \neq \emptyset, \qquad r|Z| \subset \mathrm{ext}(X) \quad \textit{for all } r > 1.$$

For the proof of the case $\delta = 1$ we refer the reader to [6]. The same argument can be applied for the case $\delta = -1$, too. See also [1] for the case $\delta = -1$ and $V_1^-(x) = V_\tau^-(z) = 1$. The foregoing proposition is the key to the next theorem on uniqueness and absence of periodic orbits of (1).

Proposition 2.4. *Let $\delta \in \{-1, 1\}$ be fixed. For every $k \in \mathbb{N} \setminus \{0\}$ there exists $\tau(k, \delta) > 0$ such that for every fixed $\tau > \tau(k, \delta)$ there exists, up to time translation, a unique periodic solution x of* (1) *for which* *(i)* $V_\tau^-(x) = 2k - 1$ *if the feedback is negative,*

(ii) $V_\tau^+(x) = 2k$ *if the feedback is positive.*

Proof. Uniqueness is contained in [5, 6]. For the case $\delta = -1$, existence can be found, e.g., in [9]. For the case $\delta = 1$, existence is given in [8], or the technique applied in Proposition 3.1 of [4], combined with the Poincaré–Bendixson theorem of Mallet-Paret and Sell [12] can give a solution whose ω-limit set is the required periodic orbit. □

Let τ^* denote $\tau(1,1)$.

Definition 2.5. For each $\tau > \tau^*$, let $T(\tau)$ denote the minimal period of the unique solution guaranteed by Proposition 2.4 in case $\delta = 1$ and $k = 1$. $T\colon (\tau^*, \infty) \to \mathbb{R}_+$ is called the *period function* for equation (1) with $\delta = 1$.

3 The period function and periodic orbits

Proposition 3.1. *Let* $\delta \in \{-1, 1\}$, $0 < \tau_1 < \tau_2$ *and let* ε *denote the sign of* δ. *Assume that* x_i, $i \in \{1, 2\}$, *is a nonconstant periodic solution of*

$$\dot{x}(t) = -\mu x(t) + \delta f(x(t - \tau_i))$$

with minimal period $T_{x_i} > 0$. *Let* X_1 *and* X_2 *denote the D-trajectories of* x_1 *and* x_2*, respectively. If* $V_{\tau_1}^\varepsilon(x_1) = V_{\tau_2}^\varepsilon(x_2) > 0$*, then*

$$|X_2| \subset \mathrm{ext}(X_1) \cup |X_1| \quad \textit{and} \quad |X_2| \cap \mathrm{ext}(X_1) \neq \emptyset.$$

Proof. In part a) we show that $|X_2| \subset \mathrm{ext}(X_1) \cup |X_1|$ and in part b) prove that $|X_2| \cap \mathrm{ext}(X_1) \neq \emptyset$.

a) Let $u_1(t) = x_1(\tau_1 t)$ and $u_2(t) = x_2(\tau_1 t)$ for all $t \in \mathbb{R}$. Then u_1 and u_2 are nonconstant periodic solutions of

$$\dot{u}_1(t) = -\mu_1 u_1(t) + \delta f_1(u_1(t-1)) \quad \text{and} \quad \dot{u}_2(t) = -\mu_1 u_2(t) + \delta f_1(u_2(t - \tau)),$$

respectively, where $\mu_1 = \tau_1 \mu, \tau = \frac{\tau_2}{\tau_1}$ and $f_1\colon \mathbb{R} \to \mathbb{R}$ is defined by $f_1(\xi) = \tau_1 f(\xi)$ for all $\xi \in \mathbb{R}$. Let U_1 and U_2 denote the D-trajectories of u_1 and u_2, respectively. Note that $|U_i|$ is a vertical scaling of $|X_i|$ by τ_1 for $i \in \{1, 2\}$.

Now, assume to the contrary that $|X_2| \cap \mathrm{int}(X_1) \neq \emptyset$. By the note above $|U_2| \cap \mathrm{int}(U_1) \neq \emptyset$ also holds, thus there exists $\beta > 1$ such that

$$\beta |U_2| \subset \mathrm{ext}(U_1) \cup |U_1| \quad \text{and} \quad \beta|U_2| \cap |U_1| \neq \emptyset.$$

Finally let $w_2(t) = \beta u_2(t)$ for all $t \in \mathbb{R}$. Then w_2 is a nonconstant periodic solution of $\dot{w}_2(t) = -\mu_1 w_2(t) + \delta g_1(w_2(t - \tau))$, where $g_1\colon \mathbb{R} \to \mathbb{R}$ is defined by $g_1(\xi) = \beta f_1(\frac{\xi}{\beta})$ for all $\xi \in \mathbb{R}$. For the D-trajectory W_2 of w_2,

$$|W_2| \subset \mathrm{ext}(U_1) \cup |U_1| \quad \text{and} \quad |W_2| \cap |U_1| \neq \emptyset$$

hold and it is easy to check that $g_1'(0) = f_1'(0)$, $V_1^\varepsilon(v_1) = V_\tau^\varepsilon(w_2) > 0$. We claim that

$$g_1(\xi) > f_1(\xi) \quad \text{and} \quad \frac{g_1'(\xi)}{g_1(\xi)} > \frac{f_1'(\xi)}{f_1(\xi)} \quad \text{for all} \quad \xi > 0,$$

which contradicts Proposition 2.3 and proves that $|X_2| \subset \operatorname{ext}(X_1) \cup |X_1|$. To prove the above claim, recall that $\xi \mapsto \frac{\xi f_1'(\xi)}{f_1(\xi)}$ is strictly monotone decreasing on $(0,\infty)$. Using $\lim_{\xi\to 0} \xi f_1'(\xi)/f_1(\xi) = 1$ we infer

$$\frac{\xi f_1'(\xi)}{f_1(\xi)} < 1 \quad \text{for all} \quad \xi > 0.$$

For every $\xi > 0$, the function $(0,\infty) \ni u \mapsto u f_1(\frac{\xi}{u}) \in \mathbb{R}$ is strictly increasing since its derivatives at $u > 0$ are given by

$$f_1\big(\frac{\xi}{u}\big)\big(1 - \frac{(\xi/u) f_1'(\xi/u)}{f_1(\xi/u)}\big) > 0.$$

This fact and $\beta > 1$ combined imply $g_1(\xi) = \beta f_1(\xi/\beta) > f_1(\xi)$ for all $\xi > 0$. Using again that $\xi \mapsto \frac{\xi f_1'(\xi)}{f_1(\xi)}$ is strictly decreasing we obtain

$$\frac{g_1'(\xi)}{g_1(\xi)} = \frac{1}{\xi}\frac{(\xi/\beta) f_1'(\xi/\beta)}{f_1(\xi/\beta)} > \frac{1}{\xi}\frac{\xi f_1'(\xi)}{f_1(\xi)} = \frac{f_1'(\xi)}{f_1(\xi)}.$$

b) Assume to the contrary that $|X_2| \cap \operatorname{ext}(X_1) = \emptyset$. According to part a), $|X_2| = |X_1|$ follows. Proposition 2.2 (i) and (ii) yield that there exist constants $c, a_1, a_2 \in \mathbb{R}$ such that

$$\begin{aligned}
\min_{t\in\mathbb{R}} x_1(t) &= x_1(a_1) = -c = x_2(a_2) = \min_{t\in\mathbb{R}} x_2(t),\\
\max_{t\in\mathbb{R}} x_1(t) &= x_1(a_1 + T(\tau_1)/2) = c = x_2(a_2 + T(\tau_2)/2) = \max_{t\in\mathbb{R}} x_2(t),\\
\dot{x}_1(a_1) &= \dot{x}_1(a_1 + T(\tau_1)/2) = \dot{x}_2(a_2) = \dot{x}_2(a_2 + T(\tau_2)/2) = 0.
\end{aligned}$$

We may assume that $a_1 = a_2 = 0$. Let x_1^{-1} and x_2^{-1} denote the inverses of the functions

$$[0, T(\tau_1)/2] \ni t \mapsto x_1(t) \in \mathbb{R} \quad \text{and} \quad [0, T(\tau_2)/2] \ni t \mapsto x_2(t) \in \mathbb{R},$$

respectively. Then the domain of x_1^{-1} and x_2^{-1} is $[-c, c]$. From $|X_1| = |X_2|$ we infer that the functions

$$\phi_1 \colon [-c,c] \ni u \mapsto \dot{x}_1(x_1^{-1}(u)) \in \mathbb{R} \quad \text{and} \quad \phi_2 \colon [-c,c] \ni u \mapsto \dot{x}_2(x_2^{-1}(u)) \in \mathbb{R}$$

satisfy $\phi_1(u) = \phi_2(u)$ for all $u \in [-c,c]$ and $\phi_1(u) > 0, \phi_2(u) > 0$ for all $u \in (-c,c)$. Thus on one hand we have

$$\int_{-c}^{c} \frac{\mathrm{d}u}{\phi_1(u)} = \int_{-c}^{c} \frac{\mathrm{d}u}{\phi_2(u)}.$$

On the other hand

$$\int_{-c}^{c} \frac{\mathrm{d}u}{\phi_i(u)} = \int_{x_i(0)}^{x_i(T(\tau_i)/2)} \frac{\mathrm{d}u}{\phi_i(u)} = \int_0^{T(\tau_i)/2} \frac{\dot{x}_i(t)}{\phi_i(x_i(t))}\mathrm{d}t = \frac{T(\tau_i)}{2}$$

holds for $i \in \{1,2\}$, and thus $T(\tau_1) = T(\tau_2)$. Let $T = T(\tau_1) = T(\tau_2)$. Now for any $b \in (-c,c)$, let $t_1(b) = x_1^{-1}(b)$ and $t_2(b) = x_2^{-1}(b)$. Then

$$\begin{aligned}
t_1(b) &= \int_0^{t_1(b)} \frac{\dot{x}_1(t)}{\phi_1(x_1(t))}\mathrm{d}t = \int_{-c}^{x_1(t_1(b))} \frac{\mathrm{d}u}{\phi_1(u)}\\
&= \int_{-c}^{x_2(t_2(b))} \frac{\mathrm{d}u}{\phi_2(u)} = \int_0^{t_2(b)} \frac{\dot{x}_2(t)}{\phi_2(x_2(t))}\mathrm{d}t = t_2(b).
\end{aligned}$$

Using Proposition 2.2 again

$$(x_1(t), \dot{x}_1(t)) = (x_2(t), \dot{x}_2(t))$$

follows for all $t \in \mathbb{R}$. Thus

$$\begin{aligned}-\mu x_1(t) + \delta f(x_1(t-\tau_1)) = \dot{x}_1(t) = \dot{x}_2(t) &= -\mu x_2(t) + \delta f(x_2(t-\tau_2)) \\ &= -\mu x_1(t) + \delta f(x_1(t-\tau_2))\end{aligned}$$

holds for all $t \in \mathbb{R}$. By the monotonicity of f we obtain $x_1(t-\tau_1) = x_1(t-\tau_2)$ for all $t \in \mathbb{R}$. Hence there exists $m \in \mathbb{N}$ such that $\tau_2 = \tau_1 + mT$. From $V^{\pm}_{\tau_1}(x_1) = V^{\pm}_{\tau_2}(x_2)$ we get $m = 0$ and thus $\tau_1 = \tau_2$ which contradicts the assumption $\tau_1 < \tau_2$. This contradiction completes our proof. □

Theorem 3.2. *The period function T of* (1) *with positive feedback satisfies*

$$0 \leq T(\tau_2) - T(\tau_1) < 2(\tau_2 - \tau_1)$$

for all $\tau_1, \tau_2 \in (\tau^*, \infty)$ with $\tau_1 < \tau_2$.

Proof. Let $\tau^* < \tau_1 < \tau_2$. Suppose that x_i is the unique periodic solution of

$$\dot{x}(t) = -\mu x(t) + f(x(t-\tau_i)), \quad i \in \{1,2\},$$

such that $V_{\tau_i}(x_i) = 2$. Let X_i be the D-trajectory of x_i for $i \in \{1,2\}$. According to Proposition 3.1 we have

$$|X_2| \subset |X_1| \cup \operatorname{ext}(X_1) \quad \text{and} \quad |X_2| \cap \operatorname{ext}(X_1) \neq \emptyset. \tag{3}$$

In part a) we prove that T is monotone nondecreasing, then in part b) we show that $T(\tau_2) - T(\tau_1) < 2(\tau_2 - \tau_1)$ for all $\tau^* < \tau_1 < \tau_2$.

a) Assume that $T(\tau_2) < T(\tau_1)$. Since, for $i \in \{1,2\}$,

$$\dot{x}_i(t) = -\mu x_i(t) + f(x_i(t-\tau_i)) = -\mu x_i(t) + f(x_i(t - (\tau_i + l \cdot T(\tau_i))))$$

holds for every $l \in \mathbb{N}$, thus x_i is a nonconstant periodic solution of

$$\dot{x} = -\mu x(t) + f(x(t - (\tau_i + l \cdot T(\tau_i)))), \quad l \in \mathbb{N},$$

too, and by Proposition 2.2, $V^+_{\tau_1 + lT(\tau_1)}(x_1) = V^+_{\tau_2 + lT(\tau_2)}(x_2) = 2(l+1)$ for all $l \in \mathbb{N}$. Since $T(\tau_2) < T(\tau_1)$, thus there exists $l^* \in \mathbb{N}$ such that $\tau_2 + l^* \cdot T(\tau_2) < \tau_1 + l^* \cdot T(\tau_1)$. Again, by Proposition 3.1 we have

$$|X_1| \subset |X_2| \cup \operatorname{ext}(X_2) \quad \text{and} \quad |X_1| \cap \operatorname{ext}(X_2) \neq \emptyset, \tag{4}$$

which is a contradiction to (3). This contradiction proves that T is monotone nondecreasing.

b) Now assume to the contrary that $T(\tau_2) - T(\tau_1) \geq 2(\tau_2 - \tau_1)$. Since f is odd and the solutions have a special symmetry, we have

$$\begin{aligned}\dot{x}_i(t) &= -\mu x_i(t) + f(x_i(t-\tau_i)) = -\mu x_i(t) + f(-x_i(t - (\tau_i - T(\tau_i)/2))) = \\ &= -\mu x_i(t) - f(x_i(t - (\tau_i - T(\tau_i)/2)))\end{aligned}$$

for $i \in \{1,2\}$. Hence, for $i \in \{1,2\}$, x_i is a nonconstant periodic solution of

$$\dot{x}(t) = -\mu x(t) - f(x(t - (\tau_i - T(\tau_i)/2)))$$

too, and by Proposition 2.2 it is clear that $\tau_i - T(\tau_i)/2 > 0$ for $i \in \{1,2\}$ and

$$V^-_{\tau_1 - T(\tau_1)/2}(x_1) = V^-_{\tau_2 - T(\tau_2)/2}(x_2).$$

If $T(\tau_2) - T(\tau_1) > 2(\tau_2 - \tau_1)$, then $\tau_2 - \frac{T(\tau_2)}{2} < \tau_1 - \frac{T(\tau_1)}{2}$, thus Proposition 3.1 leads to

$$|X_1| \subset |X_2| \cup \operatorname{ext}(X_2) \quad \text{and} \quad |X_1| \cap \operatorname{ext}(X_2) \neq 0, \tag{5}$$

which contradicts (3). Therefore $T(\tau_2) - T(\tau_1) > 2(\tau_2 - \tau_1)$ cannot occur.

If $T(\tau_2) - T(\tau_1) = 2(\tau_2 - \tau_1)$, then x_1 and x_2 are both nonconstant periodic solutions of the same delay differential equation:

$$\dot{x}(t) = -\mu x(t) - f(x(t - \tau)),$$

where $\tau = \tau_1 - T(\tau_1)/2 = \tau_2 - T(\tau_2)/2$. Then by the uniqueness of such nonconstant periodic solutions, we obtain $|X_1| = |X_2|$, which contradicts (3). This contradiction proves that $T(\tau_2) - T(\tau_1) < 2(\tau_2 - \tau_1)$, and therefore the proof is complete. □

4 Application: periodic orbits of a delayed system

Consider the system of delay differential equations

$$\begin{aligned} \dot{x}^0(t) &= -\mu x^0(t) + f(x^1(t)), \\ &\vdots \\ \dot{x}^{n-1}(t) &= -\mu x^{n-1}(t) + f(x^n(t)), \\ \dot{x}^n(t) &= -\mu x^n(t) + \delta f(x^0(t-1)), \end{aligned} \tag{6}$$

where μ, δ and f have the same properties as in equation (1) and $2 \leq n \in \mathbb{N}$.

For a system of the form (6) the natural phase space (according to [11]) is $C(\mathbb{K}, \mathbb{R})$, where $\mathbb{K} = [-1, 0] \cup \{1, 2, \ldots, n\}$. We shall use the shorter form: $C(\mathbb{K})$.

If x is a solution of (6) on some interval, then we let $x_{t,\mathbb{K}} \in C(\mathbb{K})$ be defined by

$$x_{t,\mathbb{K}}(\theta) = \begin{cases} x^0(t+\theta) & \text{for} \quad \theta \in [-1, 0], \\ x^\theta(t) & \text{for} \quad \theta \in \{1, 2, \ldots, n\}, \end{cases}$$

where it makes sense. Again, according to [11] we define now Lyapunov functionals similarly as before:

$$V_{\mathbb{K}}^+ \colon C(\mathbb{K}) \setminus \{0\} \to \{0, 2, 4, \ldots, \infty\}, \qquad V_{\mathbb{K}}^- \colon C(\mathbb{K}) \setminus \{0\} \to \{1, 3, 5, \ldots, \infty\},$$

by

$$V_{\mathbb{K}}^+(\varphi) = \begin{cases} \operatorname{sc}(\varphi, \mathbb{K}) & \text{if} \quad \operatorname{sc}(\varphi, \mathbb{K}) \quad \text{is even or infinite}, \\ \operatorname{sc}(\varphi, \mathbb{K}) + 1 & \text{if} \quad \operatorname{sc}(\varphi, \mathbb{K}) \quad \text{is odd}, \end{cases}$$

$$V_{\mathbb{K}}^-(\varphi) = \begin{cases} \operatorname{sc}(\varphi, \mathbb{K}) & \text{if} \quad \operatorname{sc}(\varphi, \mathbb{K}) \quad \text{is odd or infinite}, \\ \operatorname{sc}(\varphi, \mathbb{K}) + 1 & \text{if} \quad \operatorname{sc}(\varphi, \mathbb{K}) \quad \text{is even}, \end{cases}$$

where $\operatorname{sc}(\varphi, \mathbb{K})$ denotes the sign changes of φ just as before. The following proposition is the analogue of Proposition 2.2 and is also based on results in [12].

Proposition 4.1. *Let $x \colon \mathbb{R} \to \mathbb{R}^{n+1}$ be a nonconstant periodic solution of (6) with minimal period $T_x > 0$, and for each $j \in \{0, 1, \ldots, n\}$ let X^j denote the D-trajectory of the coordinate-function x^j. Then the following statements are true.*

(i) For each $j \in \{0,1,\dots,n\}$, X^j is a simple closed curve.

(ii) For each $j \in \{0,1,\dots,n\}$, there are $t_0^j \in \mathbb{R}$ and $t_1^j \in (t_0^j, t_0^j + T_x)$ such that $0 < \dot{x}^j(t)$ for all $t_0^j < t < t_1^j$, $x^j(\mathbb{R}) = [x^j(t_0^j), x^j(t_1^j)]$, and $\dot{x}(t) < 0$ for all $t_1^j < t < t_0^j + T_x$.

(iii) If x^0 has a zero, then $x(t+\frac{T_x}{2}) = -x(t)$ for all $t \in \mathbb{R}$. Moreover, $0 \in \operatorname{int}(X^j)$ for all $j \in \{0,1,\dots,n\}$.

(iv) There exists $k \in \mathbb{N}$ such that

$$V_{\mathbb{K}}^{+}(x_{t,\mathbb{K}}) = 2k \quad \text{for all} \quad t \in \mathbb{R} \quad \text{if} \quad \delta = 1,$$
$$V_{\mathbb{K}}^{-}(x_{t,\mathbb{K}}) = 2k+1 \quad \text{for all} \quad t \in \mathbb{R} \quad \text{if} \quad \delta = -1.$$

(v) For every nonconstant periodic solution $y\colon \mathbb{R} \to \mathbb{R}^{n+1}$ of (6) with minimal period $T_y > 0$ and $y_{t,\mathbb{K}} \neq x_{s,\mathbb{K}}$ for all $t, s \in \mathbb{R}$ we have $|Y^j| \cap |X^j| = \emptyset$ for all $j \in \{0,1,\dots,n\}$, where Y^j denotes the D-trajectory corresponding to y^j.

Just as before in the case of equation (1), if x is a solution of (6), then we shall write $V_{\mathbb{K}}^{\pm}(x)$ instead of $V_{\mathbb{K}}^{\pm}(x_{t,\mathbb{K}})$ for all $t \in \mathbb{R}$. The following two propositions from [14] are not hard to prove but they play an essential role in the sequel.

Proposition 4.2. *Assume that $x\colon \mathbb{R} \to \mathbb{R}^{n+1}$ is a solution of (6). Then $y = (y^0, \dots, y^n)\colon \mathbb{R} \ni t \mapsto (x^1(t), \dots, x^n(t), \delta x^0(t-1)) \in \mathbb{R}^{n+1}$ is also a solution of (6).*

Proposition 4.3. *Let $x\colon \mathbb{R} \to \mathbb{R}^{n+1}$ be a nonconstant periodic solution of (6) with minimal period $T_x > 0$.*

(i) If the feedback is positive and $V_{\mathbb{K}}^{+}(x) \geq 2$, then there exists $\alpha_+ \in [0, T_x)$ such that $(x^1(t), \dots, x^n(t), x^0(t-1)) = (x^0(t+\alpha_+), \dots, x^n(t+\alpha_+))$ for all $t \in \mathbb{R}$.

(ii) If the feedback is negative, then there exists $\alpha_- \in [0, T_x)$ such that $(x^1(t), \dots, x^n(t), -x^0(t-1)) = (x^0(t+\alpha_-), \dots, x^n(t+\alpha_-))$ for all $t \in \mathbb{R}$.

Proof. The argument exposed in [14] for $\delta = 1$ applies for both cases. □

We also need the following lemma, which guarantees that 0 is contained in the interior of the D-trajectory of the coordinate-functions of periodic solutions.

Lemma 4.4. *Suppose that x is a nonconstant periodic solution of (6).*

(i) In the case of negative feedback for every $j \in \{0,1,\dots,n\}$, x^j has sign changes on each interval (t^*, ∞) for all $t^* \in \mathbb{R}$.

(ii) If the feedback is positive, then exactly one of the following statements is true:

a) *$x^j(t) > 0$ for all $j \in \{0,1,\dots,n\}$ and $t \in \mathbb{R}$;*

b) *$x^j(t) < 0$ for all $j \in \{0,1,\dots,n\}$ and $t \in \mathbb{R}$;*

c) *for each $j \in \{0,1,\dots,n\}$, x^j has a zero.*

Proof. (i) Let us assume to the contrary that there exists $j \in \{0, 1, \dots, n\}$ such that $0 \le x^j(t)$ for all $t \in \mathbb{R}$ (the case $0 \ge x^j(t)$ is similar). By Proposition 4.2 we may assume that $j = 0$. Since the equation $\dot{x}^n(t) = -\mu x^n(t) - f(x^0(t-1))$ holds for all $t \in \mathbb{R}$, we have

$$\dot{x}^n(t) \le -\mu x^n(t) \text{ for all } t \in \mathbb{R}. \tag{7}$$

Since x^n is periodic, there exists $t^* \in \mathbb{R}$ such that $\dot{x}^n(t^*) = 0$. Now, from inequality (7) we obtain $x^n(t^*) \le 0$. Again, by inequality (7) $x^n(t) \le x^n(t^*)e^{\mu(t^*-t)} \le 0$ for all $t^* < t$ follows. Using the periodicity of x^n we have that $x^n(t) \le 0$ for all $t \in \mathbb{R}$. By induction we obtain that $x^i(t) \le 0$ for all $t \in \mathbb{R}$ and $i \in \{n-1, n-2, \dots, 0\}$. This together with the assumption $x^0(t) \ge 0$ for all $t \in \mathbb{R}$ means that $x^0(t) = 0$ for all $t \in \mathbb{R}$, which contradicts the fact that x is a nonconstant periodic solution of (6). This contradiction proves our claim.

(ii) See [14]. □

Remark 4.5. By the method of steps it is obvious that for any given $\varphi \in C([-\tau, 0], \mathbb{R})$ there exists exactly one solution $x\colon [-\tau, \infty) \to \mathbb{R}$ of (1) that satisfies $x|_{[-\tau,0]} = \varphi$. It is also clear that in the case $\delta = 1$, if $\varphi \in C([-\tau, 0], \mathbb{R}_+)$ and $\varphi \ne 0$, then we have $x(t) > 0$ for all large t.

The following proposition was proven in [14] only for the case of positive feedback and $n = 2$.

Proposition 4.6. *Let x be a nonconstant periodic solution of* (6) *with minimal period T_x.*

(i) In the case of positive feedback, if $\alpha_+ \in [0, T_x)$ and $k \in \mathbb{N}$ are such that $(x^1(t), \dots, x^n(t), x^0(t-1)) = (x^0(t+\alpha_+), \dots, x^n(t+\alpha_+))$ and $V^+_{\mathbb{K}}(x) = 2k+2$, then

$$\alpha_+ = \frac{(k+1)T_x - 1}{n+1}.$$

(ii) In the case of negative feedback, if $\alpha_- \in [0, T_x)$ and $k \in \mathbb{N}$ are such that $(x^1(t), \dots, x^n(t), -x^0(t-1)) = (x^0(t+\alpha_-), \dots, x^n(t+\alpha_-))$ and $V^-_{\mathbb{K}}(x) = 2k+1$, then

$$\alpha_- = \frac{(k+\frac{1}{2})T_x - 1}{n+1}.$$

Proof. The proof for the two cases are similar, hence we only prove (i). First, note that

$$x^0(t-1) = x^n(t+\alpha_+) = x^{n-1}(t+2\alpha_+) = \dots = x^0(t+(n+1)\alpha_+) \quad \text{for all} \quad t \in \mathbb{R}.$$

From Proposition 4.3 it follows that $x^i(t) = x^i(t+1+(n+1)\alpha_+)$ holds for all $i \in \{0, 1, \dots, n\}$. Thus there exists $m \in \mathbb{N}$ such that $mT_x = 1 + (n+1)\alpha_+$ or equivalently:

$$\alpha_+ = \frac{mT_x - 1}{n+1} \quad \text{for an appropriate} \quad m \in \mathbb{N}.$$

We claim that $m = k+1$. Note the following:

$$\begin{aligned}\dot{x}^0(t) &= -\mu x^0(t) + f(x^1(t)) = -\mu x^0(t) + f(x^0(t+\alpha_+)) \\ &= -\mu x^0(t) + f(x^0(t-(T_x-\alpha_+))).\end{aligned}$$

We claim that

$$\alpha_+ < \frac{T_x}{2}. \tag{8}$$

If this is not the case, then $(T_x-\alpha_+)\in(0,T_x/2)$. By Proposition 4.1, there exists t_0 such that x^0 has no zeros on the interval $[t_0,t_0+T_x-\alpha_+]$. Remark 4.5 implies that x^0 has no zeros on $\mathbb{R}$, either. Then, from Lemma 4.4 we obtain $V^+_{\mathbb{K}}(x)=0$, a contradiction which proves the claim. It follows from Proposition 2.2 that x^0 cannot be constant on any interval and hence we may choose t^* such that

$$x_{t^*,\mathbb{K}}(j)=x^j(t^*)=x^0(t^*+j\alpha_+)\neq 0 \quad \text{for any} \quad j\in\{1,2,\dots,n\}. \tag{9}$$

By combining (8) and (9) with Proposition 4.1 (i), (ii) and $V^+_{\mathbb{K}}(x)=2k+2$, we obtain that

$$\operatorname{sc}(x_{t^*,\mathbb{K}},\mathbb{K})=\operatorname{sc}(x^0,[t^*-1,t^*+n\alpha_+])\in\{2k+1,2k+2\}.$$

From this and the special symmetry of the solutions we have

$$\big(k+\frac{1}{2}\big)T_x<n\alpha_++1\le(k+1)T_x.$$

Using $\alpha_+=\frac{mT_x-1}{n+1}$ we obtain

$$(n+1)\big(k+\frac{1}{2}\big)T_x-1<nmT_x\le(n+1)(k+1)T_x-1.$$

One can see that for given $n,k\in\mathbb{N}$ and $T_x>0$ there is only one $m\in\mathbb{N}$ for which the above inequalities hold. By rearrangement we obtain the following two inequalities for m:

$$\frac{(k+\frac{1}{2})T_x-1}{n}<\frac{mT_x-1}{n+1} \tag{10}$$

and

$$\frac{mT_x-1}{n+1}\le\frac{(k+1)T_x-1}{n}. \tag{11}$$

Now it suffices to show that for $m=k+1$ both (10) and (11) hold. Let us assume to the contrary that this is not the case. Since (11) trivially holds for $m=k+1$ we obtain that (10) fails to be true, thus we have

$$\frac{(k+\frac{1}{2})T_x-1}{n}\ge\frac{(k+1)T_x-1}{n+1}$$

or equivalently

$$\big(k-\frac{n-1}{2}\big)T_x\ge 1. \tag{12}$$

a) If $\left(k-\frac{n-1}{2}\right)T_x>1$, then by an appropriate choice of $t^*\in\mathbb{R}$, we can assume that $\operatorname{sc}(x^0,[t^*-1,t^*])\le 2k-n$. Note that

$$\operatorname{sc}(x_{t^*,\mathbb{K}},\mathbb{K})\le\operatorname{sc}(x^0,[t^*-1,t^*])+n=2k$$

which is a contradiction to $V^+_{\mathbb{K}}(x)=2k+2$, thus $\left(k-\frac{n-1}{2}\right)T_x>1$ cannot occur and therefore $\left(k-\frac{n-1}{2}\right)T_x=1$.

b) If $\left(k-\frac{n-1}{2}\right)T_x=1$ and (10) fails, then it also follows that $m\ge k+2$, thus

$$\frac{(k+2)T_x-1}{n+1}\le\frac{(k+1)T_x-1}{n}$$

must hold, which is equivalent to

$$1\le(k+1-n)T_x.$$

Now using $n \geq 2$ and (12) we obtain

$$1 \leq (k+1-n)T_x = \underbrace{\big(k - \frac{n-1}{2}\big)T_x}_{=1} + \underbrace{\big(\frac{n-1}{2} - (n-1)\big)T_x}_{<0} < 1,$$

which is a contradiction. This contradiction proves the claim that $m = k+1$ and thus the proof is complete. □

Suppose that x is a nonconstant periodic solution of (6) with $\delta = 1$ and with minimal period T_x and $k \in \mathbb{N}$ is such that $V^+_{\mathbb{K}}(x) = 2k+2$. By using first Proposition 4.3 and then Proposition 4.6 and (8) we obtain

$$\begin{aligned}\dot{x}^0(t) &= -\mu x^0(t) + f(x^1(t)) = -\mu x^0(t) + f(x^0(t+\alpha_+)) \\ &= -\mu x^0(t) + f(x^0(t-(T_x-\alpha_+))) = -\mu x^0(t) + f(x^0(t-\tau_+))\end{aligned}$$

for all $t \in \mathbb{R}$, where $\tau_+ = \frac{(n-k)T_x+1}{n+1} \in (\frac{T_x}{2}, T_x]$. Thus x^0 is a nonconstant periodic solution of $\dot{x}(t) = -\mu x(t) + f(x(t-\tau_+))$ with $V^+_{\tau_+}(x^0) = 2$ and with a strong relationship between τ_+ and $T(\tau_+)$:

$$\tau_+ = \frac{(n-k)T(\tau_+)+1}{n+1}.$$

Analogously, if x is a nonconstant periodic solution of (6) with $\delta = -1$ and with minimal period T_x and $k \in \mathbb{N}$ is such that $V^-_{\mathbb{K}}(x) = 2k+1$, then x^0 is a solution of $\dot{x}(t) = -\mu x(t) + f(x(t-\tau_-))$ with $V^+_{\tau_-}(x^0) = 2$ and

$$\tau_- = \frac{(n-k+\frac{1}{2})T(\tau_-)+1}{n+1}.$$

Conversely, it is easy to check that if $k \in \mathbb{N}$ and y is a nonconstant periodic solution of (1) with $\delta = 1$ such that $V^+_\tau(y) = 2$ and $\tau = \frac{(n-k)T(\tau)+1}{n+1}$ hold, then

$$\begin{aligned}x_+(t) &= (x^0_+(t), \dots, x^j_+(t), \dots, x^n_+(t)) \\ &= \big(y(t), \dots, y\big(t + j \cdot \frac{(k+1)T(\tau)-1}{n+1}\big), \dots, y\big(t + n \cdot \frac{(k+1)T(\tau)-1}{n+1}\big)\big)\end{aligned}$$

is a nonconstant periodic solution of (6) with $\delta = 1$. From a similar argument as in the proof of Proposition 4.6 we obtain $V^+_{\mathbb{K}}(x_+) = 2k+2$. Analogously

$$\begin{aligned}x_-(t) &= (x^0_-(t), \dots, x^j_-(t), \dots, x^n_-(t)) \\ &= \big(y(t), \dots, y\big(t + j \cdot \frac{(k+\frac{1}{2})T(\tau)-1}{n+1}\big), \dots, y\big(t + n \cdot \frac{(k+\frac{1}{2})T(\tau)-1}{n+1}\big)\big)\end{aligned}$$

is a nonconstant periodic solution of (6) with $\delta = -1$ and $V^-_{\mathbb{K}}(x_-) = 2k+1$. The above argument leads to our next theorem.

Theorem 4.7. *There is a bijection between the nonconstant periodic solutions of* (6) *with* $V^+_{\mathbb{K}}(x) = 2k+2$ *and* $\delta = 1$ *and the intersections of the following two curves*

$$(\tau^*, \infty) \ni \tau \mapsto (\tau, T(\tau)) \quad \text{and} \quad \mathbb{R} \ni \zeta \mapsto \big(\frac{(n-k)\zeta+1}{n+1}, \zeta\big).$$

Analogously, there is a bijection between the nonconstant periodic solutions of (6) *with* $V^-_{\mathbb{K}}(x) = 2k+1$ *and* $\delta = -1$ *and the intersections of the curves*

$$(\tau^*, \infty) \ni \tau \mapsto (\tau, T(\tau)) \quad \text{and} \quad \mathbb{R} \ni \zeta \mapsto \big(\frac{(n-k+\frac{1}{2})\zeta+1}{n+1}, \zeta\big).$$

Lemma 4.8. *The following statements hold:*

(i) $\tau^* = \dfrac{2\pi - \arccos(\frac{\mu}{f'(0)})}{\sqrt{f'(0)^2 - \mu^2}}$ *and* $\lim_{\tau \to \tau^*+} T(\tau) = \dfrac{2\pi}{\sqrt{f'(0)^2 - \mu^2}}$,

(ii) $\lim_{\tau \to \infty} \dfrac{T(\tau)}{\tau} = 1.$

Proof. Statement (i) can be obtained by using the local Hopf bifurcation theorem [3] and a simple calculation for the value of τ^*. To prove statement (ii), let us recall that if x is the unique periodic solution of

$$\dot{x}(t) = -\mu x(t) + f(x(t-\tau)),$$

with $V_\tau^+(x) = 2$, then x is also the unique periodic solution of the equation

$$\dot{x}(t) = -\mu x(t) - f(x(t - (\tau - T(\tau)/2))),$$

with $V^-_{\tau - T(\tau)/2}(x) = 1$. Mallet-Paret and Nussbaum [10] showed in case of negative feedback that, for analogously defined period function, the quotient in statement (ii) tends to 2 as $\tau \to \infty$. Combining these we obtain that

$$\lim_{\tau \to \infty} \frac{T(\tau)}{\tau - T(\tau)/2} = 2,$$

which is equivalent to our claim. □

Theorem 4.9. *In the positive feedback case the following statements hold.*

(i) If $\mathbb{N} \ni k \geq \frac{n-1}{2}$ *and*

$$\frac{(2k+2)\pi - (n+1)\arccos\frac{\mu}{f'(0)}}{\sqrt{f'(0)^2 - \mu^2}} < 1,$$

then there exists a unique periodic solution x *of* (6) *with* $V_{\mathbb{K}}^+(x) = 2k+2$. *Otherwise there is no such solution.*

(ii) If $\mathbb{N} \ni k < \frac{n-1}{2}$ *and*

$$\frac{(2k+2)\pi - (n+1)\arccos\frac{\mu}{f'(0)}}{\sqrt{f'(0)^2 - \mu^2}} < 1,$$

then there exists a periodic solution x *of* (6) *with* $V_{\mathbb{K}}^+(x) = 2k+2$.

Proof. First we prove statement (i). We distinguish three different cases according to the value of k.

a) If $k > n$, then by Theorem 3.2 and Theorem 4.7 we obtain that the inequality

$$\frac{2\pi}{\sqrt{f'(0)^2 - \mu^2}} < \frac{(n+1)\tau^* - 1}{n-k} \tag{13}$$

is a necessary and sufficient condition for the existence and uniqueness of nonconstant periodic solutions of (6) with $V_{\mathbb{K}}^+ = 2k+2$. After using statement (i) of Lemma 4.8 and by a rearrangement of (13) we obtain the desired form.

b) If $k = n$, then again by Theorem 3.2 and Theorem 4.7 we obtain that the inequality

$$\tau^* < \frac{1}{n+1} \tag{14}$$

is a necessary and sufficient condition for the existence and uniqueness of nonconstant periodic solutions of (6) with $V_{\mathbb{K}}^{+} = 2k+2$. Using again statement (i) of Lemma 4.8 the desired form is obtained.

c) If $\frac{n-1}{2} \le k < n$, then by Theorem 3.2 and Theorem 4.7 we obtain that the inequality

$$\frac{2\pi}{\sqrt{f'(0)^2-\mu^2}} > \frac{(n+1)\tau^*-1}{n-k} \tag{15}$$

is a necessary and sufficient condition for the existence and uniqueness of nonconstant periodic solutions of (6) with $V_{\mathbb{K}}^{+} = 2k+2$. Using again statement (i) of Lemma 4.8 the desired form is obtained and so statement (i) is proven.

Statement (ii) follows from Theorem 3.2, Theorem 4.7 and statement (i) of Lemma 4.8. □

By the same argument we obtain an analogous theorem for the negative feedback case.

Theorem 4.10. *If the feedback is negative, then the following two statements hold.*

(i) Let $\mathbb{N} \ni k \ge \frac{n}{2}$. *If*

$$\frac{(2k+1)\pi-(n+1)\arccos\frac{\mu}{f'(0)}}{\sqrt{f'(0)^2-\mu^2}} < 1,$$

then there exists a unique periodic solution x *of* (6) *with* $V_{\mathbb{K}}^{-}(x) = 2k+1$. *Otherwise there is no such solution.*

(ii) Let $\mathbb{N} \ni k < \frac{n}{2}$. *If*

$$\frac{(2k+1)\pi-(n+1)\arccos\frac{\mu}{f'(0)}}{\sqrt{f'(0)^2-\mu^2}} < 1,$$

then there exists a periodic solution x *of* (6) *with* $V_{\mathbb{K}}^{-}(x) = 2k+1$.

From Theorem 4.7 and the theorem above it is clear that in order to prove statement (i) of Theorems 4.9 and 4.10 for all $k \in \mathbb{N}$ it is sufficient to prove that

Conjecture 4.11. $(\tau^*, \infty) \ni \tau \mapsto T(\tau)/\tau$ is a monotone nonincreasing function of τ.

Using the idea of Theorem 1.2 in [13], $T(\tau)/\tau \ne 4/3$ can be shown for all $\tau > \tau^*$. Then by the continuity of T and that $\lim_{\tau\to\tau^*+} T(\tau)/\tau = \frac{2\pi}{2\pi-\arccos\frac{\mu}{f'(0)}} < 4/3$, we obtain that $T(\tau)/\tau < 4/3$ for all $\tau > \tau^*$. Since the conjecture can be formulated as

$$\frac{T(\tau_2)-T(\tau_1)}{\tau_2-\tau_1} \le \frac{T(\tau_1)}{\tau_1} \text{ for all } \tau_1, \tau_2 \in (\tau^*, \infty),\ \tau_1 < \tau_2,$$

hence it is obvious that Conjecture 4.11 is a stronger statement than what we have already proven in Theorem 3.2.

An equivalent formulation of the above conjecture is that the period function of the delay differential equation

$$\dot{x}(t) = \tau\left(-\mu x(t) + f(x(t-1))\right)$$

is a monotone nonincreasing function of τ. Numerical simulations are in a good agreement with this conjecture, too.

As a confirmation of our conjecture we mention that if the feedback function is chosen to be the sign-function, which can be regarded as a limit of sigmoid functions with whom we have dealt in the whole paper, then Conjecture 4.11 can be proven.

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