To appear in Anz. Österreich. Akad. Wiss. Math.-Natur. Kl.

THOMAS' FAMILY OF THUE EQUATIONS OVER IMAGINARY QUADRATIC FIELDS. II

CLEMENS HEUBERGER, ATTILA PETHŐ, AND ROBERT F. TICHY

ABSTRACT. We completely solve the family of relative Thue equations

$$x^3 - (t-1)x^2y - (t+2)xy^2 - y^3 = \mu,$$

where the parameter t, the root of unity μ and the solutions x and y are integers in the same imaginary quadratic number field. This is achieved using the hypergeometric method for $|t| \geq 53$ and Baker's method combined with a computer search using continued fractions for the remaining values of t.

Let F be an irreducible form of degree at least 3 with integral coefficients and m be a nonzero integer. Then the Diophantine equation

$$F(x, y) = m$$

is called a *Thue* equation in honor of Thue [10] who proved that it has only finitely many solutions over the integers. Algorithms for solving single Thue equations over $\mathbb{Z}$ have been developed, see Bilu and Hanrot [1].

Starting with Thomas [9] in 1990, several families of parametrized Thue equations (of positive discriminant) have been solved, cf. the surveys [4, 3].

In the last years, a few parametrized families of relative Thue equations where the parameter and the solutions are elements of an imaginary quadratic number field have been studied by the authors [6], by Ziegler [11, 12], and by Jadrijević and Ziegler [7].

In [6], the parametrized family of Thue equations

$$(1) \qquad x^3 - (t-1)x^2y - (t+2)xy^2 - y^3 = \mu, \qquad x, y \in \mathbb{Z}_{\mathbb{Q}(t)},\ t \text{ imaginary quadratic integer}, \quad \mu \text{ a root of unity in } \mathbb{Z}_{\mathbb{Q}(t)}$$

has been studied. This is the family that Thomas [9] and Mignotte [8] solved completely in the rational integer case. In [6], all solutions for $|t| > 3.023 \cdot 10^9$ have been found using Baker's method. Furthermore, all solutions for $\operatorname{Re} t = -1/2$ were claimed to be listed. However, the proof of [6, Theorem 3] is incorrect (more precisely, the argument for excluding the possibility $\Lambda = 0$ in [6, Section 7] is invalid) and some solutions are missing in [6, Table 2].

By combining the hypergeometric method due to Thue and Siegel (for values $|t| \geq 53$) and lower bounds for linear forms in logarithms ("Baker's method") together with a computer search (using continued fraction expansions) for $|t| < 53$, the Diophantine equation (1) can be solved *completely* for *all values of* t.

The details are discussed in the forthcoming paper [2]. The purpose of this note is to announce the corrected and complete result:

The first and the third author were supported by the Austrian Science Foundation FWF, projects S9606 and S9603, respectively, that are part of the Austrian National Research Network "Analytic Combinatorics and Probabilistic Number Theory." Research was partly done during a visit of the first and the third author at the Department of Computer Science of the University of Debrecen in the frame of a joint Austrian-Hungarian project granted by the Austrian Exchange Service ÖAD (No. A-27/2003) and the Hungarian Tét foundation. They thank the Department for its hospitality. Other parts were done during a visit of the first author at the Institute of Mathematics of the University of Zagreb in the frame of a joint Austrian-Croatian project granted by the Austrian Exchange Service ÖAD (No. 20/2004 and 23/2006) and the Croatian Ministry of Science, Education and Sports. He thanks the institute for its hospitality. The second author was partially supported by Hungarian National Foundation for Scientific Research Grant No. T42985.

Theorem. *Let t be an integer in an imaginary quadratic number field, $t \notin \{(-1 \pm 3\sqrt{-3})/2\}$, $\mathbb{Z}_{\mathbb{Q}(t)}$ be the ring of integers of $\mathbb{Q}(t)$,*

$$F_t(X,Y) = X^3 - (t-1)X^2Y - (t+2)XY^2 - Y^3 \in \mathbb{Z}_{\mathbb{Q}(t)}[X,Y],$$

and μ be a root of unity in $\mathbb{Q}(t)$.

Then all solutions $(x,y) \in \mathbb{Z}^2_{\mathbb{Q}(t)}$ to

$$F_t(x,y) = \mu \tag{2}$$

are listed in Table 1 (solutions independent of t) and in the online Table [5] *(solutions for specific values of t). A short summary of these 732 "sporadic" solutions is given in Table 2. The sporadic solutions with* $\operatorname{Re} t = -1/2$ *are listed in Table 3.*

x	y	μ	x	y	μ	x	y	μ
0	1	-1	i	0	$-i$	$-1+\omega_3$	$1-\omega_3$	-1
-1	0	-1	0	i	i	ω_3	0	-1
1	-1	-1	$-i$	0	i	0	$1-\omega_3$	1
0	-1	1	i	$-i$	i	0	ω_3	1
-1	1	1	0	$-\omega_3$	-1	$-\omega_3$	0	1
1	0	1	0	$-1+\omega_3$	-1	$1-\omega_3$	$-1+\omega_3$	1
0	$-i$	$-i$	$-\omega_3$	ω_3	-1	$-1+\omega_3$	0	1
$-i$	i	$-i$	$1-\omega_3$	0	-1	ω_3	$-\omega_3$	1

TABLE 1. Solutions (if contained in $\mathbb{Q}(t)$) to (2) for all t, where $\omega_3 = (1+\sqrt{-3})/2$.

Remark. If $t \in \{(-1 \pm 3\sqrt{-3})/2\}$ then $F_t(X,Y)$ is the cube of a linear polynomial. Thus (2) has infinitely many solutions (x,y) for all roots of unity $\mu \in \mathbb{Q}(\sqrt{-3})$ in this case.

Acknowledgment. The authors thank Volker Ziegler for pointing out the mistakes in their original paper [6].

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t	Number of solutions	$\max\{\lvert x\rvert^2, \lvert y\rvert^2\}$
-4	6	81
-2	6	9
-1	12	81
0	12	81
1	6	9
3	6	81
$-1 \pm 2i$	24	5
$-1 \pm 3i$	24	5
$\pm 2i$	24	5
$\pm 3i$	24	5
$-1 \pm \sqrt{-2}$	6	9
$-1 \pm 2\sqrt{-2}$	6	3
$\pm\sqrt{-2}$	6	9
$\pm 2\sqrt{-2}$	6	3
$-2 \pm 2\sqrt{-3}$	12	688
$(-3 \pm 3\sqrt{-3})/2$	24	7
$-1 \pm \sqrt{-3}$	24	3
$-1 \pm 2\sqrt{-3}$	6	1
$(-1 \pm \sqrt{-3})/2$	18	27
$\pm\sqrt{-3}$	24	3
$\pm 2\sqrt{-3}$	6	1
$(1 \pm 3\sqrt{-3})/2$	24	7
$1 \pm 2\sqrt{-3}$	12	688
$-2 \pm \sqrt{-5}$	6	86
$1 \pm \sqrt{-5}$	6	86
$-1 \pm \sqrt{-7}$	12	4
$(-1 \pm \sqrt{-7})/2$	6	7
$\pm\sqrt{-7}$	12	4
$(-3 \pm \sqrt{-11})/2$	6	20
$(1 \pm \sqrt{-11})/2$	6	20
$(-1 \pm \sqrt{-19})/2$	6	19
$(-1 \pm \sqrt{-31})/2$	6	98
$(-1 \pm \sqrt{-35})/2$	6	611

TABLE 2. Overview on sporadic solutions to (2) for specific t.

(C. Heuberger and R. F. Tichy) INSTITUT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT GRAZ, STEYRERGASSE 30, A-8010 GRAZ, AUSTRIA

E-mail address: clemens.heuberger@tugraz.at, tichy@tugraz.at

(A. Pethő) DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF DEBRECEN, P.O. BOX 12, H-4010 DEBRECEN, HUNGARY

E-mail address: pethoe@inf.unideb.hu

t	x	y
$(-1 \pm \sqrt{-3})/2$	$\pm 3\sqrt{-3}$	$(1 \pm 3\sqrt{-3})/2$
$(-1 \pm \sqrt{-3})/2$	$(-5 \pm \sqrt{-3})/2$	$-2 \pm \sqrt{-3}$
$(-1 \pm \sqrt{-3})/2$	$(5 \pm \sqrt{-3})/2$	$(-9 \pm 3\sqrt{-3})/2$
$(-1 \pm \sqrt{-3})/2$	$-2 \pm \sqrt{-3}$	$(9 \pm 3\sqrt{-3})/2$
$(-1 \pm \sqrt{-3})/2$	$2 \pm \sqrt{-3}$	$(5 \pm \sqrt{-3})/2$
$(-1 \pm \sqrt{-3})/2$	$(-9 \pm 3\sqrt{-3})/2$	$2 \pm \sqrt{-3}$
$(-1 \pm \sqrt{-3})/2$	$(-1 \pm 3\sqrt{-3})/2$	$\pm 3\sqrt{-3}$
$(-1 \pm \sqrt{-3})/2$	$(1 \pm 3\sqrt{-3})/2$	$(-1 \pm 3\sqrt{-3})/2$
$(-1 \pm \sqrt{-3})/2$	$(9 \pm 3\sqrt{-3})/2$	$(-5 \pm \sqrt{-3})/2$
$(-1 \pm \sqrt{-7})/2$	$\pm\sqrt{-7}$	$(-1 \pm \sqrt{-7})/2$
$(-1 \pm \sqrt{-7})/2$	$(-1 \pm \sqrt{-7})/2$	$(1 \pm \sqrt{-7})/2$
$(-1 \pm \sqrt{-7})/2$	$(1 \pm \sqrt{-7})/2$	$\pm\sqrt{-7}$
$(-1 \pm \sqrt{-19})/2$	$\pm\sqrt{-19}$	$(-3 \pm \sqrt{-19})/2$
$(-1 \pm \sqrt{-19})/2$	$(-3 \pm \sqrt{-19})/2$	$(3 \pm \sqrt{-19})/2$
$(-1 \pm \sqrt{-19})/2$	$(3 \pm \sqrt{-19})/2$	$\pm\sqrt{-19}$
$(-1 \pm \sqrt{-31})/2$	$\pm\sqrt{-31}$	$(-19 \pm \sqrt{-31})/2$
$(-1 \pm \sqrt{-31})/2$	$(-19 \pm \sqrt{-31})/2$	$(19 \pm \sqrt{-31})/2$
$(-1 \pm \sqrt{-31})/2$	$(19 \pm \sqrt{-31})/2$	$\pm\sqrt{-31}$
$(-1 \pm \sqrt{-35})/2$	$\pm 2\sqrt{-35}$	$24 \pm \sqrt{-35}$
$(-1 \pm \sqrt{-35})/2$	$-24 \pm \sqrt{-35}$	$\pm 2\sqrt{-35}$
$(-1 \pm \sqrt{-35})/2$	$24 \pm \sqrt{-35}$	$-24 \pm \sqrt{-35}$

TABLE 3. Sporadic solutions to $F_t(x, y) = 1$ for $\operatorname{Re} t = -1/2$. The solutions to $F_t(x, y) = -1$ are the negatives of the listed values. There are no solutions to $F_t(x, y) = \mu$ for roots of unity μ other than for $\mu \in \{-1, 1\}$ for $\operatorname{Re} t = -1/2$.